

First order logic of permutations

Michael Albert, Mathilde Bouvel and Valentin Féray

June 28, 2016

PP2017 (Reykjavik University)



What is a permutation?

- ▶ An element of some group G acting on a finite set X ?
- ▶ A bijective map $f : X \rightarrow X$ for some (finite) set X ?
- ▶ A bijective map $f : [n] \rightarrow [n]$ ($[n] = \{1, 2, \dots, n\}$)?
- ▶ A word of length n from the alphabet $[n]$ without repeated letters?
- ▶ The result of taking n i.i.d. samples from a permutation?
- ▶ A finite set X equipped with two linear orders *by position* ($\leq_P$) and *by value* ($\leq_V$)?

Depending on your answer, the language and logic you use to discuss permutations will change (as will the questions that you will tend to ask).

The theory of two orders (TOTO) is the framework of the final answer – it is axiomatised by sentences that require the two relations $\leq_P$ and $\leq_V$ be linear orders, i.e.,

$$\forall x \forall y \quad x \leq y \vee y \leq x$$

$$\forall x \forall y \quad x \leq y \wedge y \leq x \Rightarrow x = y$$

$$\forall x \forall y \forall z \quad x \leq y \wedge y \leq z \Rightarrow x \leq z$$

(with each subscript.)

What can be said?

Lots of stuff!

- ▶ I begin with my maximum value:

$$\exists x \forall y \quad x \leq_P y \wedge x \geq_V y$$

- ▶ I avoid 312 (with obvious generalisations)

$$\forall x \forall y \forall z \quad x \leq_P y \leq_P z \Rightarrow \neg (y \leq_V z \leq_V x)$$

- ▶ A formula which is satisfied by a if a is a *cut-point* of the permutation:

$$CP(x) := \forall y \quad (y \leq_P x \wedge y \leq_V x) \vee (x \leq_P y \wedge x \leq_V y)$$

What can be said?

Lots of stuff!

- ▶ I begin with my maximum value:

$$\exists x \forall y \quad x \leq_P y \wedge x \geq_V y$$

- ▶ I avoid 312 (with obvious generalisations)

$$\forall x \forall y \forall z \quad x \leq_P y \leq_P z \Rightarrow \neg (y \leq_V z \leq_V x)$$

- ▶ A formula which is satisfied by a if a is a *cut-point* of the permutation:

$$CP(x) := \forall y \quad (y \leq_P x \wedge y \leq_V x) \vee (x \leq_P y \wedge x \leq_V y)$$

- ▶ Avoiding/involving mesh patterns

What can be said?

Lots of stuff!

- ▶ I begin with my maximum value:

$$\exists x \forall y \quad x \leq_P y \wedge x \geq_V y$$

- ▶ I avoid 312 (with obvious generalisations)

$$\forall x \forall y \forall z \quad x \leq_P y \leq_P z \Rightarrow \neg (y \leq_V z \leq_V x)$$

- ▶ A formula which is satisfied by a if a is a *cut-point* of the permutation:

$$CP(x) := \forall y \quad (y \leq_P x \wedge y \leq_V x) \vee (x \leq_P y \wedge x \leq_V y)$$

- ▶ Avoiding/involving mesh patterns
- ▶ I am plus (minus) (in)decomposable, or simple.
- ▶ etc.

The puffin-hole principle



Image source: <https://baldmonkeyseenabird.wordpress.com/tag/puffins/>

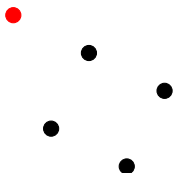
The puffin-hole principle

- ▶ Two permutations are *TOTO- k -equivalent* ($\equiv_k$) if they satisfy the same sentences of quantifier depth k in TOTO.
- ▶ Since, up to renaming of variables, there are only finitely many such sentences, there are lots of different permutations that are TOTO- k -equivalent (too many puffins).
- ▶ This suggests a strategy for finding things we can't say in TOTO:
 - ▶ **Given**: a property P of permutations,
 - ▶ **Found**: distinct TOTO- k -equivalent permutations ω (witness) and λ (liar) such that ω satisfies P ($\omega \models P$) but λ does not,
 - ▶ **Conclusion**: P cannot be defined using k or fewer quantifiers.
- ▶ So how do we recognise TOTO- k -equivalence?

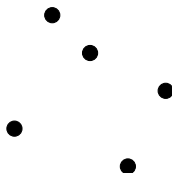
Meet the contestants

- ▶ *Spoiler* believes that $\pi \not\equiv_k \sigma$, *Duplicator* believes $\pi \equiv_k \sigma$.
- ▶ Who is correct?
- ▶ They agree to play a game consisting of k rounds.
- ▶ In each round Spoiler chooses an element of either π or σ , and then Duplicator chooses an element of the other permutation (repeated choices are allowed).
- ▶ At the end of the game we have a sequence $(p_1, p_2, \dots, p_k)$ of elements of π and $(s_1, s_2, \dots, s_k)$ of σ .
- ▶ Duplicator wins if the assignment $p_i \mapsto s_i$ is an order-preserving isomorphism (in particular, $p_i = p_j$ if and only if $s_i = s_j$).
- ▶ Whoever wins the game (assuming the stakes were high enough) was right!

Duplicator loses in two!



52413



25413

How can Duplicator respond?

- ▶ If she chooses the greatest element, 5, Spoiler follows with 2 which is to the left.
- ▶ If she chooses any other element, Spoiler responds with 5.
- ▶ So, she loses regardless.
- ▶ Not coincidentally “I begin with my greatest element” was expressible by a sentence of depth two.

Duplicator wins in three!



- ▶ If Spoiler's first move is "near" one end or the other (two or fewer points beyond the move), Duplicator replies in the same position relative to the end of the other permutation.
- ▶ If he plays in the middle of the first permutation, or in the middle three positions of the second, she responds similarly in the middle of the other permutation.
- ▶ If his next move is in a "short" (two or fewer points) segment, she responds in the corresponding one at the corresponding place.
- ▶ If he plays in a "long" segment, she mimics her first move strategy (but "near" now means one or fewer points).

Fixed points

Proposition

There is no TOTO-formula $FP(x)$ such that $\pi \models FP(a)$ if and only if a is a fixed point of π , nor is there a TOTO-sentence FP such that $\pi \models FP$ if and only if π has a fixed point.

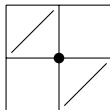
- ▶ A decreasing permutation has a fixed point if and only if it is of odd size.
- ▶ But, for any k all sufficiently long decreasing permutations are $\equiv_k$ equivalent.
- ▶ The “formula” case is really just a slight extension of the game (basically the elements named by the formula are pre-set before the game begins).

Moving the goal posts

Question

In which permutation classes $\mathcal{C}$ is there a TOTO-formula (sentence) defining fixed points?

From the preceding result, $\mathcal{C}$ must avoid at least one decreasing pattern. Suppose though that $\mathcal{C}$ contains 321 and consider application of the magic lemma to $321[\mathcal{I}, 1, \mathcal{I}]$:



The central dot is a fixed point if and only if the two segments have the same size. So, if both segments can become arbitrarily large we're out of luck. Thus there must be a pattern of the form $321[\mathcal{I}, 1, \mathcal{I}]$ that is not in the class.

Is that enough?

- ▶ For convenience assume that neither the decreasing permutation δ_{k+2} nor $321[\iota_{k+1}, 1, \iota_{k+1}]$ are in $\mathcal{C}$.
- ▶ Suppose that a is a fixed point of π in $\mathcal{C}$. That means that there are equal number of elements “above and left” and “below and right” of a .
- ▶ But, if there were $k^2 + 1$ or more in both regions we would not be in $\mathcal{C}$.
- ▶ So we can define:

$FP_{\mathcal{C}}(x) :=$ for some $t \leq k^2$ there are exactly t elements in the two significant regions relative to x

Stable subpermutations

- ▶ A *stable occurrence* of σ in π is an occurrence of σ as a pattern in π which is also a union of orbits of π .
- ▶ E.g., a stable occurrence of 1 is a fixed point, a stable occurrence of 21 is any 2-cycle in π , the 3-cycles in π are the stable occurrences of 231 or 312.

Theorem

A permutation class $\mathcal{C}$ admits a formula $\text{Stab}_{\mathcal{C}}^{\sigma}(\mathbf{x})$ such that $\pi \models \text{Stab}_{\mathcal{C}}^{\sigma}(\mathbf{a})$ means that $\mathbf{a}$ is a stable occurrence of σ in π if and only if $\mathcal{C}$ avoids at least one permutation in each of a finite explicit list of classes.

In this case the “sentence” case is a bit different. There is no formula identifying stable of occurrences of 21 in a decreasing permutation but there is a sentence (‘the permutation has at least three elements’)

Some other things we know

- ▶ Broadly speaking “sorting classes” are all TOTO-definable (e.g., 17-stack sortable in the sense of West). Moreover the definitions can be recovered “automatically”.
- ▶ We can characterize exactly the sets of permutations that are both TOTO and BUS (Bijection of an Unordered Set) definable.
- ▶ (with Marc Noy) First order convergence laws for **some** classical pattern classes. Note these must be convergence laws rather than 0-1 laws since for example “I begin with my minimum element” has asymptotic probability $1/4$ in $\text{Av}(321)$.

What we don't know

Lots of things!

- ▶ For which σ are “formula-definability” and “sentence-definability” of stable occurrences of σ the same? (Conjecture: for all σ except decreasing permutations of even size.)
- ▶ Applications of more general versions of the magic lemmata (or whether these might not be necessary).
- ▶ How small can a permutation class that contains all cycle types be? More generally, what criteria on permutation classes are sufficient to ensure that all cycle types occur?
- ▶ First order convergence in general?
- ▶ Are we still in Kansas?

Finally

Thank you

**The America's Cup
is once again
New Zealand's cup!**



Image source: <https://www.stuff.co.nz>